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On some new results for non-decreasing sequences

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Abstract

In this paper, a general theorem on absolute Riesz summability factors of infinite series is proved under weaker conditions. Also we have obtained some new and known results.

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1 Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (s_n), that is (see [6]),

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \texttt{for} \quad n > 0. \tag{2}$$

A series $\sum a_n$ is said to be summable $\mid C, \alpha; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [8])

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k < \infty. \tag{3}$$

If we take δ=0, then we obtain $\mid C, \alpha \mid_k$ summability (see [7]). Let (p_n) be a sequence of positive numbers such that $P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1)$. The sequence-to-sequence transformation

$$v_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{4}$$

defines the sequence (v_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n),generated by the sequence of coefficients (p_n) (see [9]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [3])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} \mid v_n - v_{n-1} \mid^k < \infty. \tag{5}$$

If we take δ=0, the we obtain $\mid \bar{N}, p_n \mid_k$ summability (see [1]). In the special case $p_n = 1$ for all values of n $\mid \bar{N}, p_n; \delta \mid_k$ summability is the same as $\mid C, 1; \delta \mid_k$ summability. Also if we take $\delta = 0$

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and $k=1$, then we get $\mid \bar{N}, p_n \mid$ summability.

2. Known results. The following theorems are known dealing with $\mid \bar{N}, p_n \mid_k$ and $\mid \bar{N}, p_n; \delta \mid_k$ summability factors of infinite series.

Theorem A ([2]). Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (β_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{6}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{7}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{8}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{9}$$

If

$$\sum_{n=1}^{m} \frac{\mid s_n \mid^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{11}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{12}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

Theorem B ([4]). Let (X_n) be a positive non-decreasing sequence. If the sequences (X_n), (β_n), (λ_n), and (p_n) satisfy the conditions (6)-(9), (11)-(12), and

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{\mid s_n \mid^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{13}$$

$$\sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} = O((\frac{P_v}{p_v})^{\delta k} \frac{1}{P_v}) \quad as \quad m \to \infty, \tag{14}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

Remark. It should be noted that if we take $\delta = 0$, then we get Theorem A. In this case condition (13) reduces to condition (10) and condition (14) reduces to

$$\sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = \sum_{n=v+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) = O\left(\frac{1}{P_v}\right) \quad as \quad m \to \infty, \tag{15}$$

which always holds. Also it may be noticed that, under the conditions on the sequence (λ_n) we have that (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [2]).

3. Main result. The aim of this paper is to prove Theorem B under weaker conditions. Now, we shall prove the following theorem.

Theorem. Let (X_n) be a positive non-decreasing sequence. If the sequences (X_n), (β_n), (λ_n), and (p_n) satisfy the conditions (6)-(9), (11)-(12), (14), and

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{\mid s_n \mid^k}{n X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{16}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

Remark. It should be noted that condition (16) is the same as condition (13) when k=1. When $k > 1$, condition (16) is weaker than condition (13) but the converse is not true. As in [12], we can show that if (13) is satisfied, then we get

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{\mid s_n \mid^k}{n X_n^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{\mid s_n \mid^k}{n} = O(X_m).$$

To show that the converse is false when $k > 1$, as in [5], the following example is sufficient. We can take $X_n = n^\sigma$, $0 < \sigma < 1$, and then construct a sequence (u_n) such that

$$u_n = (\frac{P_n}{p_n})^{\delta k} \frac{\mid s_n \mid^k}{n X_n^{k-1}} = X_n - X_{n-1},$$

hence

$$\sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k} \frac{\mid s_n \mid^k}{n X_n^{k-1}} = X_m = m^\sigma,$$

and so

$$\begin{aligned} \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{\mid s_n \mid^k}{n} &= \sum_{n=1}^{m} (X_n - X_{n-1}) X_n^{k-1} = \sum_{n=1}^{m} (n^\sigma - (n-1)^\sigma) n^{\sigma(k-1)} \\ &\geq \sigma \sum_{n=1}^{m} n^{\sigma-1} n^{\sigma(k-1)} = \sigma \sum_{n=1}^{m} n^{\sigma k - 1} \sim \frac{m^{\sigma k}}{k} \quad as \quad m \to \infty. \end{aligned}$$

It follows that

$$\frac{1}{X_m} \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{\mid s_n \mid^k}{n} \to \infty \quad as \quad m \to \infty$$

provided $k > 1$. This shows that (13) implies (16) but not conversely.
We require the following lemmas for the proof of our theorem.

Lemma 1.1([10]). Under the conditions on (X_n), (β_n) and (λ_n) as as expressed in the statement of the theorem, we have the following ;

$$n X_n \beta_n = O(1), \tag{17}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{18}$$

Lemma 3.2 ([11]). If the conditions (11) and (12) are satisfied, then $\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right)$.

4. Proof of the theorem. Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{np_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{r p_r} = \frac{1}{P_n}\sum_{v=1}^{n}(P_n - P_{v-1})\frac{a_v P_v \lambda_v}{v p_v}.$$

Then we get that

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{v p_v}, \quad n \geq 1, \quad (P_{-1} = 0).$$

By using Abel's transformation, we have that

$$\begin{aligned}
T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} s_v \Delta\left(\frac{P_{v-1} P_v \lambda_v}{v p_v}\right) + \frac{\lambda_n s_n}{n} \\
&= \frac{s_n \lambda_n}{n} + \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} s_v \frac{P_{v+1} P_v \Delta\lambda_v}{(v+1)p_{v+1}} \\
&+ \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v s_v \lambda_v \Delta\left(\frac{P_v}{v p_v}\right) - \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} s_v P_v \lambda_v \frac{1}{v} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} | T_{n,r} |^k < \infty, \quad \texttt{for} \quad r = 1,2,3,4. \tag{19}$$

Applying Abel's transformation, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} | T_{n,1} |^k &= \sum_{n=1}^{m}\left(\frac{P_n}{np_n}\right)^{k-1}\left(\frac{P_n}{p_n}\right)^{\delta k} | \lambda_n |^{k-1} | \lambda_n | \frac{| s_n |^k}{n} \\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{| s_n |^k}{n}\left(\frac{1}{X_n}\right)^{k-1} | \lambda_n | \\
&= O(1)\sum_{n=1}^{m-1}\Delta | \lambda_n | \sum_{v=1}^{n}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{| s_v |^k}{v {X_v}^{k-1}} \\
&+ O(1) | \lambda_m | \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{| s_n |^k}{n {X_n}^{k-1}}
\end{aligned}$$

$$= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m$$
$$= O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1), \quad as \quad m \to \infty,$$

by the hypotheses of the theorem and Lemma 3.1. Now, by using (12) and applying Hölder's inequality we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,2} \mid^k = O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k} \mid \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v \mid^k$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v} \mid s_v \mid p_v \mid \Delta\lambda_v \mid\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k p_v {\beta_v}^k$$
$$\times \left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right)^{k-1}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k p_v {\beta_v}^k \sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} {\beta_v}^{k-1}\beta_v\left(\frac{P_v}{p_v}\right)^{\delta k} \mid s_v \mid^k$$
$$= O(1)\sum_{v=1}^{m}(v\beta_v)^{k-1}\beta_v\left(\frac{P_v}{p_v}\right)^{\delta k} \mid s_v \mid^k$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{1}{X_v}\right)^{k-1}\beta_v\left(\frac{P_v}{p_v}\right)^{\delta k} \mid s_v \mid^k$$
$$= O(1)\sum_{v=1}^{m} v\beta_v\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{\mid s_v \mid^k}{v{X_v}^{k-1}}$$
$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{\mid s_r \mid^k}{r{X_r}^{k-1}}$$
$$+ \quad O(1) \quad m\beta_m\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{\mid s_v \mid^k}{v{X_v}^{k-1}}$$
$$= O(1)\sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1)m\beta_m X_m$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1} \mid (v+1)\Delta\beta_v - \beta_v \mid X_v + O(1)m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1)\sum_{v=1}^{m-1} X_v\beta_v + O(1)m\beta_m X_m = O(1),
\end{aligned}
$$

as $m \to \infty$, by the hypotheses of the theorem and Lemma 3.1. Again, as in $T_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,3} \mid^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_v \Delta\left(\frac{P_v}{v p_v}\right) \mid^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1} P_v \mid s_v \mid\mid \lambda_v \mid \frac{1}{v}\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right) p_v \mid s_v \mid\mid \lambda_v \mid \frac{1}{v}\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{v p_v}\right)^k p_v \mid s_v \mid^k\mid \lambda_v \mid^k \\
&\times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{v p_v}\right)^k \mid s_v \mid^k p_v \mid \lambda_v \mid^k \sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{v p_v}\right)^k p_v \mid s_v \mid^k\mid \lambda_v \mid^k \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}.\frac{v}{v} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{v p_v}\right)^{k-1} \mid \lambda_v \mid^{k-1}\mid \lambda_v \mid \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{1}{X_v}\right)^{k-1} \mid \lambda_v \mid \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v {X_v}^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1} X_v\beta_v + O(1)X_m \mid \lambda_m \mid = O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by the hypotheses of the theorem, Lemma 3.1 and Lemma 3.2. Finally, using Hölder's inequality, as in $T_{n,3}$, we have get

$$
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,4} \mid^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^k} \mid \sum_{v=1}^{n-1} s_v \frac{P_v}{v}\lambda_v \mid^k
$$

$$
\begin{aligned}
&= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k}\mid\sum_{v=1}^{n-1}s_v\frac{P_v}{vp_v}p_v\lambda_v\mid^k\\
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\mid s_v\mid^k\left(\frac{P_v}{vp_v}\right)^k p_v\mid\lambda_v\mid^k\\
&\times \left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k\left(\frac{P_v}{p_v}\right)^{\delta k}\mid s_v\mid^k p_v\mid\lambda_v\mid^k\frac{1}{P_v}.\frac{v}{v}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^{k-1}\mid\lambda_v\mid^{k-1}\mid\lambda_v\mid\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{\mid s_v\mid^k}{v}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{1}{X_v}\right)^{k-1}\mid\lambda_v\mid\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{\mid s_v\mid^k}{v}\\
&= O(1)\sum_{v=1}^{m}\mid\lambda_v\mid\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{\mid s_v\mid^k}{v{X_v}^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}X_v\beta_v+O(1)X_m\mid\lambda_m\mid=O(1)\quad as\quad m\to\infty.
\end{aligned}
$$

This completes the proof of the theorem. If we take δ=0, then we get a new result dealing with $\mid \bar{N},p_n\mid_k$ summability factors of infinite series. If we take $k=1$ and δ=0, then we get a known result of Mishra and Srivastava dealing with $\mid \bar{N},p_n\mid$ summability factors of infinite series (see [11]). Finally, if we take $p_n=1$ for all values of n, then we get a new result concerning the $\mid C,1;\delta\mid_k$ summability factors of infinite series.

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